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The effects of combined horizontal and vertical heterogeneity on the onset of convection in a porous medium

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Abstract

The effects of hydrodynamic and thermal heterogeneity, for the case of variation in both the horizontal and vertical directions, on the onset of convection in a horizontal layer of a saturated porous medium uniformly heated from below, are studied analytically for the case of weak heterogeneity. It is found that the effect of such heterogeneity on the critical value of the Rayleigh number *Ra* based on mean properties is of second order if the properties vary in a piecewise constant or linear fashion. The effects of horizontal heterogeneity and vertical heterogeneity are then comparable once the aspect ratio is taken into account, and to a first approximation are independent. For the case of conducting impermeable top and bottom boundaries and a square box, the effects of permeability heterogeneity and conductivity heterogeneity each cause a reduction in the critical value of *Ra*, while for the case of a tall box there can be either a reduction or an increase.

Keywords: Natural convection; Heterogeneity; Instability; Horton–Rogers–Lapwood problem

1. Introduction

The classical Horton–Rogers–Lapwood problem, for the onset of convection in a horizontal layer of a saturated porous medium uniformly heated from below, has been extensively studied. Studies of the effects of heterogeneity in this situation are surveyed in Nield and Bejan [1], so a comprehensive survey is not attempted here. However, we record that the pioneering study was that of Gheorghitza [2]. Particularly notable are the studies of vertical heterogeneity (especially the case of horizontal layers) by McKibbin and O'Sullivan [3,4], McKibbin and Tyvand [5–7], Nield [8] and Leong and Lai [9,10], and the studies of horizontal heterogeneity by McKibbin [11], Nield [12] and Guonot and Caltagirone [13]. Some more general aspects of conductivity heterogeneity have been discussed by Vadasz [14], Braester and Vadasz [15] and Rees and Riley [16]. However, it appears that one set of questions has been left unanswered: namely in what respects, if any, does the effect of vertical heterogeneity differ from the effect of horizontal heterogeneity for each of the permeability (hydrodynamic) and conductivity (thermal) types, and how do these two types interact with each other?

The topic of permeability heterogeneity [17] is currently of interest for an additional reason. Simmons et al. [18] and Prasad and Simmons [19] have pointed out that in many heterogeneous geologic systems [20], hydraulic properties such as the hydraulic conductivity of the system under consideration can vary by many orders of magnitude and sometimes rapidly over small spatial scales. They also pointed out that the onset of instability in transient, sharp interface problems is controlled by very local conditions in the vicinity of the evolving boundary layer and not by the global layer properties or indeed some average property of that macroscopic layer. They also correctly pointed out that any averaging process would remove the very structural controls and physics that are expected to be

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Nomenclature

A	aspect ratio (height to width)
c	specific heat
k	k^*/k_0
k^*	overall (effective) thermal conductivity
k_0	mean value of $k^*(x^*, y^*)$
K	K^*/K_0
K^*	permeability
K_0	mean value of $K^*(x^*, y^*)$
L	height (and width) of the enclosure
P	dimensionless pressure, $\frac{(\rho c)_f K_0}{\mu k_0} P^*$
P^*	pressure
R	$Ra/4\pi^2$
Ra	Rayleigh number, $\frac{(\rho c)_f \rho_0 g \beta K_0 L (T_1 - T_0)}{\mu k_0}$
t^*	time
t	dimensionless time, $\frac{k_0}{(\rho c)_m L^2} t^*$
T^*	temperature
T_0	temperature at the upper boundary
T_1	temperature at the lower boundary
u	dimensionless horizontal velocity, $\frac{(\rho c)_m L}{k_0} u^*$
$\mathbf{u}^*$	vector of Darcy velocity, (u^*, v^*)
v	dimensionless vertical velocity, $\frac{(\rho c)_m L}{k_0} v^*$
x	dimensionless horizontal coordinate, x^*/L
x^*	horizontal coordinate
y	dimensionless upward vertical coordinate, y^*/L
y^*	upward vertical coordinate

Greek symbols

β	fluid volumetric expansion coefficient
θ	dimensionless temperature, $\frac{T^* - T_0}{T_1 - T_0}$
μ	fluid viscosity
ρ	density
ρ	fluid density at temperature T_0
σ	heat capacity ratio, $\frac{(\rho c)_m}{(\rho c)_f}$
ψ	streamfunction defined by Eq. (10a,b)

Subscripts

f	fluid
m	overall porous medium

Superscript

*	dimensional variable

important in controlling the onset, growth, and/or decay of instabilities in a highly heterogeneous system. In particular, in the case of dense plume migration in highly heterogeneous environments the application of an average global Rayleigh number based upon average hydraulic conductivity of the medium is problematic. In these cases, an average Rayleigh number is unable to predict the onset of instability accurately because the system is characterized by unsteady flows and large amplitude perturbations. For this reason, it is desirable to look again at theoretical studies that lead to a prediction of the criterion for instability.

Nield and Simmonds [21] have emphasized the need to distinguish between weak heterogeneity and strong heterogeneity. For the case of weak heterogeneity (properties varying by a factor not greater than 3 or so) the introduction of an equivalent Rayleigh number is useful. The extent to which an equivalent Rayleigh number (based on averaged permeability and averaged conductivity) might work was investigated by Nield [8] for the case of vertical heterogeneity. He concluded that provided the variation of each of the various parameters lies within one order of magnitude, a rough and ready estimate of an effective Rayleigh number can be made that is useful as a criterion for Rayleigh–Bénard convection. This effective Rayleigh number is based on the arithmetic mean quantities (such as the permeability) that appear in the numerator, and the harmonic mean of quantities (such as the viscosity) that appear in the denominator of the defining expression. Similar conclusions were drawn by Leong and Lai [9,10]. In the case of strong heterogeneity the concept of an effective Rayleigh number loses validity as a criterion for the onset of instability.

In this paper, we look again at the case of weak heterogeneity for the general case involving both vertical heterogeneity and horizontal heterogeneity. For this complicated situation no exact analytical solution can be expected to exist, but it is reasonable to seek an approximate analytical solution. One would expect that for weak heterogeneity the solution would not differ dramatically from the solution for the homogeneous case. Following this approach, we utilize an extension of the Galerkin approximate method that has been widely employed (see, for example, Finlayson [22]). In the context of the onset of convection, the Galerkin method commonly used involves trial functions of the vertical coordinate only. In the analysis that follows trial functions of both the vertical and horizontal coordinates are introduced, and these trial functions are chosen to be the known exact solutions for the homogeneous case.

The problem studied here is the heterogeneous extension of the analysis for the classical Horton–Roger–Lapwood (HRL) problem for the case of impervious thermally conducting bottom and top boundaries. Two-dimensional convection in a square box with impervious thermally insulated side walls is examined. This geometry is chosen because it is well known that the favored form of convection in the homogeneous HRL problem for a horizontal layer is a pattern of cells with square cross-section and by symmetry there is no heat flux normal to the cell side walls. This means that we have in mind a horizontal layer of infinite extent, but we have effectively pre-selected the horizontal wavenumber of the disturbances in the resulting instability problem, taking it as the critical value for the homogeneous case. Thus there is little loss of generality in the choice of aspect ratio of the box as far as the horizontal layer situation is concerned. However, later in the paper the effect of a change in aspect ratio is investigated.

2. Analysis

Single-phase flow in a saturated porous medium is considered. Asterisks are used to denote dimensional variables. We consider a square box, $0 \leqslant x^* \leqslant L$, $0 \leqslant y^* \leqslant L$, where the y^* axis is in the upward vertical direction. The side walls are taken as insulated, and uniform temperatures T_0 and T_1 are imposed at the upper and lower boundaries, respectively.

Within this box the permeability is $K^*(x^*, y^*)$ and the overall (effective) thermal conductivity is $k^*(x^*, y^*)$. The Darcy velocity is denoted by $\mathbf{u}^* = (u^*, v^*)$. The Oberbeck–Boussinesq approximation is invoked. The equations representing the conservation of mass, thermal energy, and Darcy's law take the form

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{1}$$

$$(\rho c)_{\mathrm{m}} \frac{\partial T^*}{\partial t^*} + (\rho c)_{\mathrm{f}} \left[u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right] = k^*(x^*, y^*) \left[\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} \right], \tag{2}$$

$$u^* = -\frac{K^*(x^*, y^*)}{\mu} \frac{\partial P^*}{\partial x^*},$$
$$v^* = \frac{K^*(x^*, y^*)}{\mu} \left[-\frac{\partial P^*}{\partial y^*} - \rho_0 \beta g (T^* - T_0) \right]. \tag{3a, b}$$

Here $(\rho c)_{\mathrm{m}}$ and $(\rho c)_{\mathrm{f}}$ are the heat capacities of the overall porous medium and the fluid, respectively, μ is the fluid viscosity, ρ is the fluid density at temperature T_0, and β is the volumetric expansion coefficient.

We introduce dimensionless variables by defining

$$(x, y) = \frac{1}{L}(x^*, y^*), \quad (u, v) = \frac{(\rho c)_{\mathrm{m}} L}{k_0}(u^*, v^*), \quad t = \frac{k_0}{(\rho c)_{\mathrm{m}} L^2} t^*,$$
$$\theta = \frac{T^* - T_0}{T_1 - T_0}, \quad P = \frac{(\rho c)_{\mathrm{f}} K_0}{\mu k_0} P^*, \tag{4a, b, c, d, e}$$

where k_0 is the mean value of $k^*(x^*, y^*)$ and K_0 is the mean value of $K^*(x^*, y^*)$.

We also define a Rayleigh number Ra by

$$Ra = \frac{(\rho c)_{\mathrm{f}} \rho_0 g \beta K_0 L (T_1 - T_0)}{\mu k_0} \tag{5}$$

and the heat capacity ratio

$$\sigma = \frac{(\rho c)_{\mathrm{m}}}{(\rho c)_{\mathrm{f}}}. \tag{6}$$

The governing equations then take the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7}$$

$$\frac{\partial \theta}{\partial \tau} + \frac{1}{\sigma} \left[u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} \right] = k(x, y) \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right], \tag{8}$$

$$u = -K(x, y) \frac{\partial P}{\partial x}, \quad v = K(x, y) \left[-\frac{\partial P}{\partial y} + \sigma Ra \theta \right], \tag{9}$$

where $k(x, y) = k^*(x^*, y^*)/k_0$ and $K(x, y) = K^*(x^*, y^*)/K_0$.

We introduce a streamfunction ψ so that

$$u = \sigma Ra \frac{\partial \psi}{\partial y}, \quad v = -\sigma Ra \frac{\partial \psi}{\partial x}. \tag{10a, b}$$

We also eliminate P. In doing this we assume that, in accordance with the assumption of weak heterogeneity, that the maximum variation of K over the domain is small compared with the mean value of K, so we can approximate $\partial(u/\hat{K})/\partial \hat{x}$ by $(1/\hat{K})\partial u/\partial \hat{x}$, etc. The result is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -K(x, y) \frac{\partial \theta}{\partial x}, \tag{11}$$

$$\frac{\partial \theta}{\partial \tau} + Ra \left[\frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} \right] = k(x, y) \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right]. \tag{12}$$

The conduction solution is given by

$$\psi = 0, \quad \theta = 1 - y. \tag{13a, b}$$

The perturbed solution is given by

$$\psi = \varepsilon \psi', \quad \theta = 1 - y + \varepsilon \theta'. \tag{14a, b}$$

To first order in the small constant ε, we get

$$\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + K(x, y) \frac{\partial \theta'}{\partial x} = 0, \tag{15}$$

$$\frac{\partial \theta'}{\partial \tau} + Ra \frac{\partial \psi'}{\partial x} - k(x, y) \left[\frac{\partial^2 \theta'}{\partial x^2} + \frac{\partial^2 \theta'}{\partial y^2} \right] = 0. \tag{16}$$

For the onset of convection we can invoke the "principal of exchange of stabilities" and hence take the time derivative in Eq. (16) to be zero.

The boundary conditions are

$$\psi' = 0 \quad \text{and} \quad \theta' = 0 \quad \text{on } y = 0, \tag{17a, b}$$
$$\psi' = 0 \quad \text{and} \quad \theta' = 0 \quad \text{on } y = 1, \tag{18a, b}$$
$$\psi' = 0 \quad \text{and} \quad \partial\theta'/\partial x = 0 \quad \text{on } x = 0, \tag{19a, b}$$
$$\psi' = 0 \quad \text{and} \quad \partial\theta'/\partial x = 0 \quad \text{on } x = 1. \tag{20a, b}$$

This set of boundary conditions is satisfied by

$$\psi'_{mn} = \sin m\pi x \sin n\pi y, \quad m, n = 1, 2, 3, \ldots \tag{21}$$
$$\theta'_{pq} = \cos p\pi x \sin q\pi y, \quad p, q = 1, 2, 3, \ldots \tag{22}$$

We can take this set of functions (that are exact eigenfunctions for the homogeneous case) as trial functions for an approximate solution of the heterogeneous case. For example, working at second order, we can try

$$\psi' = A_{11}\psi'_{11} + A_{12}\psi'_{12} + A_{21}\psi'_{21} + A_{22}\psi'_{22}, \tag{23}$$
$$\theta' = B_{11}\theta'_{11} + B_{12}\theta'_{12} + B_{21}\theta'_{21} + B_{22}\theta'_{22}. \tag{24}$$

In the Galerkin method, the expression (23) is substituted into the left-hand side of Eq. (15) and the resulting residual is made orthogonal to the separate trial functions ψ'_{11}, ψ'_{12}, ψ'_{21}, ψ'_{22} in turn. Likewise the residual on the substitution of the expression (24) into Eq. (16) is made orthogonal to θ'_{11}, θ'_{12}, θ'_{21}, θ'_{22} in turn. We use the notation

$$\langle f(x,y)\rangle = \int_0^1 \int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y, \tag{25}$$

and define

$$I_{mnpq} = 4\langle K(x,y)\sin m\pi x \sin n\pi y \sin p\pi x \sin q\pi y\rangle, \tag{26}$$

$$J_{mnpq} = 4\langle k(x,y)\cos m\pi x \sin n\pi y \cos p\pi x \sin q\pi y\rangle. \tag{27}$$

We note that $\langle k(x,y)\rangle = 1$ and $\langle K(x,y)\rangle = 1$. Also,

$$4\langle \sin m\pi x \sin n\pi y \sin p\pi x \sin q\pi y\rangle = \begin{cases} 1 & \text{if } m=p \text{ and } n=q \\ 0 & \text{otherwise} \end{cases}, \tag{28}$$

$$4\langle \cos m\pi x \sin n\pi y \cos p\pi x \sin q\pi y\rangle = \begin{cases} 1 & \text{if } m=p \text{ and } n=q \\ 0 & \text{otherwise} \end{cases}. \tag{29}$$

The output of the Galerkin procedure is a set of 8 homogeneous linear equations in the 8 unknown constants A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, B_{21}, B_{22}. Eliminating these constants we get

$$\det \mathbf{M} = 0, \tag{30}$$

where the matrix $\mathbf{M}$ takes the form

$$\mathbf{M} = \begin{bmatrix} 2\pi^2 & 0 & 0 & 0 & \pi I_{1111} & \pi I_{1211} & 2\pi I_{2111} & 2\pi I_{2211} \\ 0 & 5\pi^2 & 0 & 0 & \pi I_{1112} & \pi I_{1212} & 2\pi I_{2112} & 2\pi I_{2212} \\ 0 & 0 & 5\pi^2 & 0 & \pi I_{1121} & \pi I_{1221} & 2\pi I_{2121} & 2\pi I_{2221} \\ 0 & 0 & 0 & 8\pi^2 & \pi I_{1122} & \pi I_{1222} & 2\pi I_{2122} & 2\pi I_{2222} \\ \pi Ra & 0 & 0 & 0 & 2\pi^2 J_{1111} & 5\pi^2 J_{1211} & 5\pi^2 J_{2111} & 8\pi^2 J_{2211} \\ 0 & \pi Ra & 0 & 0 & 2\pi^2 J_{1112} & 5\pi^2 J_{1212} & 5\pi^2 J_{2112} & 8\pi^2 J_{2212} \\ 0 & 0 & 2\pi Ra & 0 & 2\pi^2 J_{1121} & 5\pi^2 J_{1221} & 5\pi^2 J_{2121} & 8\pi^2 J_{2221} \\ 0 & 0 & 0 & 2\pi Ra & 2\pi^2 J_{1122} & 5\pi^2 J_{1222} & 5\pi^2 J_{2122} & 8\pi^2 J_{2222} \end{bmatrix} \tag{31}$$

In the general case, the integrals in Eqs. (26) and (27) can be obtained by quadrature. The eigenvalue equation, Eq. (30) can then be solved to give the critical Rayleigh number.

3. Results and discussion

3.1. First order results

For example, the order-one Galerkin method (using a single trial function for each of ψ and θ) yields the eigenvalue equation

$$\det \begin{bmatrix} 2\pi^2 & \pi I_{1111} \\ \pi Ra & 2\pi^2 J_{1111} \end{bmatrix} = 0, \tag{32}$$

which gives

$$Ra = 4\pi^2 J_{1111}/I_{1111}. \tag{33}$$

For the homogeneous case, $I_{1111} = 1$ and $J_{1111} = 1$, and so $Ra = 4\pi^2$, the well known value for the Horton–Rogers–Lapwood problem. We also observe that the same value is obtained if $K(x,y) = k(x,y)$ for all (x,y), that is if the permeability heterogeneity and the conductivity heterogeneity follow the same pattern of variation.

As a further example, consider the case of permeability heterogeneity with conductivity homogeneity. Then $J_{1111} = 1$ but

$$I_{1111} = 4\langle K(x,y)\sin^2 \pi x \sin^2 \pi y\rangle, \tag{34}$$

something that is in general different from unity.

In particular, consider a quartered square, namely the present square divided by the lines $x = 1/2$, $y = 1/2$. If K takes the values c_1, c_2, c_3, c_4 in the respective quarters, then, since $\int_0^{1/2} \sin^2 \pi x\,\mathrm{d}x = \frac{1}{2}$ and $\int_{1/2}^1 \sin^2 \pi x\,\mathrm{d}x = \frac{1}{2}$ we find that

$$I_{1111} = (c_1 + c_2 + c_3 + c_4)/4 = 1, \tag{35}$$

because the mean value of K is 1. Thus in this case the heterogeneity does not alter the critical value of the Rayleigh number based on the mean permeability at first order.

Further, it appears that a linear variation of permeability does not affect the critical value of Ra at first order. The cases $K(x,y) = 1 + c(x - 1/2)(y - 1/2)$ and $K(x,y) = 1 + c_1(x - 1/2) + c_2(y - 1/2)$ each lead to

$$I_{1111} = 1. \tag{36}$$

3.2. Second order results

In order to examine the interaction of permeability heterogeneity and conductivity heterogeneity we return to Eqs. (30) and (31) and apply these to the quartered square with piecewise-constant properties. We consider the case

$$\begin{aligned} &K(x,y) = 1 - \delta_\mathrm{H} - \delta_\mathrm{V}, \\ &k(x,y) = 1 - \varepsilon_\mathrm{H} - \varepsilon_\mathrm{V}, \quad \text{for } 0 < x < 1/2, \quad 0 < y < 1/2; \\ &K(x,y) = 1 + \delta_\mathrm{H} - \delta_\mathrm{V}, \\ &k(x,y) = 1 + \varepsilon_\mathrm{H} - \varepsilon_\mathrm{V}, \quad \text{for } 1/2 < x < 1, \quad 0 < y < 1/2; \\ &K(x,y) = 1 - \delta_\mathrm{H} + \delta_\mathrm{V}, \\ &k(x,y) = 1 - \varepsilon_\mathrm{H} + \varepsilon_\mathrm{V}, \quad \text{for } 0 < x < 1/2, \quad 1/2 < y < 1; \\ &K(x,y) = 1 + \delta_H + \delta_\mathrm{V}, \\ &k(x,y) = 1 + \varepsilon_\mathrm{H} + \varepsilon_\mathrm{V}, \quad \text{for } 1/2 < x < 1, \quad 1/2 < y < 1. \end{aligned} \tag{37}$$

This case approximates a general case in which each slowly varying quantity is approximated by a piecewise-constant distribution. The mean value of the quantity is approximated by its value at centre of the main square:

$$\bar{f} = f(0.5, 0.5).$$

In each quarter, the function is approximated by its value at the centre of that quarter, and a truncated Taylor series expansion is used to approximate this factor. For example, in the region $1/2 < x < 1$, $1/2 < y < 1$, $f(x,y)$ is approximated by $f(0.75,0.75)$ and then by $f(0.5,0.5) + 0.25f_x(0.5,0.5) + 0.25f_y(0.5,0.5)$.

Thus

$$\delta_H = \frac{1}{4}\left[\frac{1}{K}\frac{\partial K}{\partial x}\right]_{(1/2,1/2)}, \quad \delta_V = \frac{1}{4}\left[\frac{1}{K}\frac{\partial K}{\partial y}\right]_{(1/2,1/2)},$$
$$\varepsilon_H = \frac{1}{4}\left[\frac{1}{k}\frac{\partial k}{\partial x}\right]_{(1/2,1/2)}, \quad \varepsilon_V = \frac{1}{4}\left[\frac{1}{k}\frac{\partial k}{\partial y}\right]_{(1/2,1/2)}. \tag{38}$$

In terms of the shorthand notation

$$\Delta_H = (8/3\pi)\delta_H, \quad \Delta_V = (8/3\pi)\delta_V,$$
$$E_H = (8/3\pi)\varepsilon_H, \quad E_V = (8/3\pi)\varepsilon_V, \tag{39}$$

one has

$$\begin{aligned} &I_{1111} = I_{1212} = I_{2121} = I_{2222} = 1 \\ &I_{1211} = I_{1112} = I_{2221} = I_{2122} = -\Delta_H \\ &I_{2111} = I_{2212} = I_{1121} = I_{1222} = -\Delta_V \\ &I_{2211} = I_{2112} = I_{1221} = I_{1122} = 0, \end{aligned} \tag{40}$$

and

$$\begin{aligned} &J_{1111} = J_{1212} = J_{2121} = J_{2222} = 1 \\ &J_{1211} = J_{1112} = J_{2221} = J_{2122} = -E_H \\ &J_{2111} = J_{2212} = J_{1121} = J_{1222} = -E_V \\ &J_{2211} = J_{2112} = J_{1221} = J_{1122} = 0. \end{aligned} \tag{41}$$

Writing $R = Ra/4\pi^2$,

$$\alpha = 1, \quad \beta = 4/25, \quad \gamma = 16/25, \quad \delta = 1/4,$$
$$\lambda = 5/2, \quad \mu = 5/4, \quad \sigma = 4/5, \quad \tau = 8/5, \tag{42}$$

and manipulating the rows and columns of the determinant, one obtains the form

$$\det\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -\Delta_H & -\Delta_V & 0 \\ 0 & 1 & 0 & 0 & -\Delta_H & 1 & 0 & -\Delta_V \\ 0 & 0 & 1 & 0 & -\Delta_V & 0 & 1 & -\Delta_H \\ 0 & 0 & 0 & 1 & 0 & -\Delta_V & -\Delta_H & 1 \\ \alpha R & 0 & 0 & 0 & 1 & -\lambda E_H & -\mu E_V & 0 \\ 0 & \beta R & 0 & 0 & -\lambda^{-1}E_H & 1 & 0 & -\sigma E_V \\ 0 & 0 & \gamma R & 0 & -\mu^{-1}E_V & 0 & 1 & -\tau E_H \\ 0 & 0 & 0 & \delta R & 0 & -\sigma^{-1}E_V & -\tau^{-1}E_H & 1 \end{bmatrix} = 0 \tag{43}$$

This expands to give a quartic equation in R, and the smallest root is sought. For the homogeneous case this is $R = 1$.

Consider the case where Δ_H, Δ_V, E_H, and E_V are all small compared with unity.

One can now set $R = 1 + S$ where S is small compared with unity. Substituting, linearizing, and solving for S one obtains

$$S = -\frac{1}{63}[3(2\Delta_H - 5E_H)^2 + 7(4\Delta_V - 5E_V)^2]. \tag{44}$$

This leads to the critical value

$$Ra = 4\pi^2\left\{1 - \frac{64}{567\pi^2}[3(2\delta_H - 5\varepsilon_H)^2 + 7(4\delta_V - 5\varepsilon_V)^2]\right\}$$
$$\approx 40\{1 - 0.137(\delta_H - 2.5\varepsilon_H)^2 - 1.281(\delta_V - 1.25\varepsilon_V)^2\}. \tag{45}$$

A number of conclusions can be drawn. The effects of weak horizontal heterogeneity and vertical heterogeneity are each of second order in the property deviations. Their combined contribution is of the order of the variances of the distributions for permeability and conductivity (which are here equal to $\delta_H^2 + \delta_V^2$ and $\varepsilon_H^2 + \varepsilon_V^2$, respectively.) The effect of vertical heterogeneity is somewhat greater than that of horizontal heterogeneity. Further, they act independently at this order of approximation. (Product terms like $\delta_H\delta_V$ are absent in the last expression.) Since the expression in square brackets in Eq. (44) is positive definite, the heterogeneities lead to a reduction in the critical value of Ra for all combinations of horizontal and vertical heterogeneities and all combinations of permeability and conductivity heterogeneities. (The reduction is zero for the very special case where $\delta_H = 2.5\varepsilon_H$ and $\delta_V = 1.25\varepsilon_V$.) The effects of the horizontal permeability heterogeneity and the horizontal conductivity heterogeneity are at the first combination step subtractive (and similarly with horizontal replaced by vertical), as one might expect since the permeability appears in the numerator in the definition of Ra whereas the conductivity appears in the denominator.

At first sight it appears from Eq. (45) that the effect of conductivity heterogeneity is substantially greater than the effect of permeability heterogeneity. However, the expression in that equation is somewhat misleading. In this expression Ra is based on the arithmetic mean conductivity, whereas a better comparison is when the effective Rayleigh number is based on the harmonic mean conductivity, as noted by Nield [8]. The harmonic mean of $1 - \varepsilon$ and $1 + \varepsilon$ is $1 - \varepsilon^2$. When allowance is made for this, the effects of permeability heterogeneity and conductivity heterogeneity are found to be approximately equal.

4. Extensions to the analysis

4.1. Three-dimensional heterogeneity

The methodology used in this paper allows an extension to the case of three-dimensional heterogeneity in a cubical box. We have not made the effort to carry out this analysis, because we expect the result to be just the addition to Eq. (45) of additional terms involving horizontal heterogeneity, symmetrical to the existing terms involving horizontal heterogeneity.

4.2. Tall box

For a box with height-to-width aspect ratio A, where A is an integer, one can scale the x-coordinate by the factor

A. One then recovers the same equations as before but now Eqs. (15) and (16) are replaced by

$$A^2\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2}+K(x,y)A\frac{\partial\theta'}{\partial x}=0, \tag{46}$$

$$\frac{\partial\theta'}{\partial\tau}+RaA\frac{\partial\psi'}{\partial x}-k(x,y)\left[A^2\frac{\partial^2\theta'}{\partial x^2}+\frac{\partial^2\theta'}{\partial y^2}\right]=0. \tag{47}$$

That eigenvalue equation becomes

$$\det\begin{bmatrix} 1 & 0 & 0 & 0 & I_{1111} & I_{1211} & I_{2111} & I_{2211} \\ 0 & 1 & 0 & 0 & I_{1112} & I_{1212} & I_{2112} & I_{2212} \\ 0 & 0 & 1 & 0 & I_{1121} & I_{1221} & I_{2121} & I_{2221} \\ 0 & 0 & 0 & 1 & I_{1122} & I_{1222} & I_{2122} & I_{2222} \\ \alpha R & 0 & 0 & 0 & J_{1111} & \lambda J_{1211} & \mu J_{2111} & \nu J_{2211} \\ 0 & \beta R & 0 & 0 & \lambda^{-1}J_{1112} & J_{1212} & \rho J_{2112} & \sigma J_{2212} \\ 0 & 0 & \gamma R & 0 & \mu^{-1}J_{1121} & \rho^{-1}J_{1221} & J_{2121} & \tau J_{2221} \\ 0 & 0 & 0 & \delta R & \nu^{-1}J_{1122} & \sigma^{-1}J_{1222} & \tau^{-1}J_{2122} & J_{2222} \end{bmatrix}=0, \tag{48}$$

where

$$\alpha=\frac{4A^2}{(A^2+1)^2},\quad \beta=\frac{4A^2}{(A^2+4)^2},\quad \gamma=\frac{16A^2}{(4A^2+1)^2},\quad \delta=\frac{16A^2}{(A^2+1)^2},$$
$$\lambda=\frac{A^2+4}{A^2+1},\quad \mu=\frac{4A^2+1}{2A^2+2},\quad \nu=\frac{2A^2+2}{A^2+1},\quad \rho=\frac{4A^2+1}{2A^2+8},$$
$$\sigma=\frac{4A^2+4}{2A^2+8},\quad \tau=\frac{4A^2+4}{4A^2+1}. \tag{49}$$

For a quartered box with piecewise-constant distributions of K and k, Eqs. (40) and (41) still hold.

In the limit as $A\to\infty$, one gets the limiting values

$$A^2\alpha=4,\quad A^2\beta=4,\quad A^2\gamma=1,\quad A^2\delta=16,$$
$$\lambda=1,\quad \mu=2,\quad \nu=2,\quad \rho=2,\quad \sigma=2,\quad \tau=1. \tag{50}$$

One finds that

$$R/A^2=1+S \tag{51}$$

where

$$S=\frac{1}{15}\left[(16\Delta_{\rm H}-E_{\rm H})(\Delta_{\rm H}-E_{\rm H})+5(2\Delta_{\rm V}-E_{\rm V})^2\right]. \tag{52}$$

This leads to

$$Ra=4\pi^2A^2\left\{1+\frac{64}{135\pi^2}[(16\delta_{\rm H}-\varepsilon_{\rm H})(\delta_{\rm H}-\varepsilon_{\rm H})+5(2\delta_{\rm V}-\varepsilon_{\rm V})^2]\right\}$$
$$\approx 40A^2\{1+0.048(16\delta_{\rm H}-\varepsilon_{\rm H})(\delta_{\rm H}-\varepsilon_{\rm H})+0.240(2\delta_{\rm V}-\varepsilon_{\rm V})^2\}. \tag{53}$$

Comparison with Eq. (45) shows that the homogeneous case value of Ra is increased by the factor A^2, while the effect of heterogeneity is no longer monotonic. The vertical heterogeneity leads to an increase in Ra and the horizontal heterogeneity produces either an increase or decrease depending on the value of $\delta_{\rm H}/\varepsilon_{\rm H}$.

One should note that the effects of the horizontal and vertical contributions are immediately comparable only if one uses the total amount of variation across the box as a the measure of heterogeneity. If one uses the rate of variation with distance as the criterion, one has to take account of the fact that the x- and y- coordinates have been differently scaled, by a factor A. For example, in terms of quantities evaluated at the centre of the box,

$$\frac{\delta_{\rm V}}{\delta_{\rm H}}=\frac{\partial K/\partial y}{\partial K/\partial x}=\frac{H}{L}\frac{\partial K^*/\partial y^*}{\partial K^*/\partial x^*}=A\frac{\partial K^*/\partial y^*}{\partial K^*/\partial x^*}. \tag{54}$$

Thus if A is large then the vertical heterogeneity has a greater impact than the horizontal heterogeneity, other things being equal.

5. Conclusions

We have initiated a study of the relationships between the effects of horizontal and vertical heterogeneities on the onset of convection in a porous medium. For the case of weak heterogeneity we have employed an approximate analysis to reach some general conclusions. We have shown that a Rayleigh number based on mean properties is a good basis for the prediction of the onset of instability. Expressions for the critical value of this parameter in terms of measures of the heterogeneity have been obtained. It has been found that piecewise-constant or linear property variation leads to effects that enter at second order in small variations, while the effect of nonlinear property variation enters at first order. The piecewise-constant case has been investigated in detail. For this case it has been shown that the effects of horizontal heterogeneity and vertical heterogeneity are comparable and to a first approximation are independent. For the case of a square box with conducting impermeable top and bottom, the effects of permeability heterogeneity and conductivity permeability in any combination cause a reduction in the critical value of Ra. For a tall box there can be either a reduction or an increase.

The cases of moderate or strong heterogeneity remain as challenges for future work. We believe that it is likely that moderate heterogeneity can be treated by numerical methods along roughly the same lines as the present work. Strong heterogeneity may require a more radical treatment.

6. Note added in proof

This paper is concerned with weak heterogeneity, and the basic assumption is that for each property (permeability, conductivity) the maximum variation of the property over the domain is small compared with the mean value of that property. Consequently no term in $\partial k^*/\partial x^*$ or $\partial k^*/\partial y^*$ appears on the right-hand side of Eq. (2). It can be shown that this approximation has no effect on the results presented in this paper provided that k^* is a linear function of the spatial variables considered separately. A similar assumption about the variation of the permeability is made in writing Eq. (11).

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